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CONTENTS.

PAGES

On a Modification of Gibbs's Statistics for Discontinuous Distribution in Phase-space—(*Contd.*): B. N. Biswas and D. K. Sen ... ... 137—143

Surfaces whose Asymptotic Lines constitute a set of Parallels: V. Rangachariar ... 144—150

The Curve of Population: V. Jambunathan, M.A., B.Sc.... 151—155

Determinants and the Roots of an Equation: Raymond Garver ... ... 156—160

NOTES AND QUESTIONS:

India's Contribution to the Theory of Indeterminate Equations of the First Degree: Saradakanta Ganguli ... ... *153—168*

Group Theory and the History of Logarithms: G. A. Miller ... ... *169—172*

Solutions ... ... *173—175*

Questions for Solution *176*

A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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5. As a first approximation we may write, from (17),

$$\mu = \frac{n}{a} \qquad \dots \quad (18)$$

Now $a = \frac{\Delta\tau}{\varepsilon}$, because ε is the volume of each cell; and n is equal to $\rho \,.\, \Delta\tau$.

Therefore, $$\mu = \rho \,.\, \varepsilon = \frac{N \,.\, \varepsilon}{A} e^{(\psi - U)/\Theta} \qquad \dots \quad (19)$$

from (11) and (3). We next select the constant A so that

$$\frac{N \,.\, \varepsilon}{A} = 1. \qquad \dots \quad (20)$$

Therefore $$\mu = e^{(\psi - U)/\Theta}. \qquad \dots \quad (21)$$

In six-dimensional phase-space we generally take $\varepsilon = h^3$, so that $A = Nh^3$. If the general phase-space is taken into account ε would be h^{3m} and $A = Nh^{3m}$.

6. We shall now proceed to find out the distribution over the whole phase-space which may be taken to be divided into smaller regions over each of which U will be practically constant; that is to say, each of which contains R-points corresponding to an energy-range between U and $U + dU$. Such a small region will, however, include a large number of elements of volume $\Delta\tau$. Let us denote these elements (within this definite energy-domain) by $\Delta\tau_1$, $\Delta\tau_2$, etc., the numbers of cells contained in these by a_1, a_2, etc., and the numbers of R-points therein by n_1, n_2, etc. Also let c and ν denote the total number of cells and R-points within this particular domain of phase-space. Then

$$c = \Sigma\, a_r \text{ and } \nu = \Sigma\, n_r. \qquad \dots \quad (22)$$

From (18) and (21),

$$\frac{n_r}{a_r} = \mu = e^{(\psi - U)/\Theta}.$$

Therefore,

$$n_r = a_r e^{(\psi - U)/\Theta}. \qquad \dots \quad (23)$$

So,

$$\nu = \Sigma\, n_r = e^{(\psi - U)/\Theta} \Sigma\, a_r = c \,.\, e^{(\psi - U)/\Theta} \qquad \dots \quad (24)$$

6 APR 1932

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which is in accord with the distribution corresponding to the Classical Statistics.

7. Now for a further approximation, we take $\mu = \frac{n}{n+a}$, or what comes to the same thing, $\frac{n}{a} = \frac{\mu}{1-\mu}$, where μ is given by (21). Thus

$$\frac{n_r}{a_r} = \frac{\mu}{1-\mu} = \frac{1}{\mu^{-1}-1} \quad \ldots \quad (25)$$

Therefore,

$$n_r = \frac{a_r}{e^{-(\psi-U)/\Theta}-1}$$

Thus

$$\nu = \Sigma\, n^r = \frac{c}{e^{-(\psi-U)/\Theta}-1} \quad \ldots \quad (26)$$

as U is constant within this region. Therefore, the total number of R-points in the whole of phase-space is given by

$$N = \Sigma \frac{c}{e^{-(\psi-U)/\Theta}-1} \quad \ldots \quad (27)$$

the summation being taken over all such energy-domains. Or, using the notation of calculus, whenever it is admissible,

$$N = \frac{2\pi\,.\,V\,(2m)^{\frac{3}{2}}}{h^3}\int_0^\infty \frac{U^{\frac{1}{2}}\,.\,dU}{e^{-(\psi-U)/\Theta}-1} \quad \ldots \quad (28)$$

because the number c of cells is known to be given by

$$2\pi V\,(2m)^{\frac{3}{2}}\,.\,U^{\frac{1}{2}}\,.\,dU/h^3,$$

the phase-space being six-dimensional and the systems represented therein having no potential. The equation (28) can be easily seen to correspond to the distribution due to Bose and Einstein.

8. We have seen that μ is a small number. If it be sufficiently small,—as is usually the case,—we can safely neglect $a_0\mu^2$, $a_0\mu^3$, etc. in comparison with a_0 and $a_0\mu$; that is, the number of cells (within an element of volume Δ_τ) containing two or more R-points, will be very small compared with the number of empty cells, or of cells with

one R-point each.* Practically, therefore, all cells are either empty or have only one R-point. This is the statistics put forward by Fermi and Dirac.

It follows that in this case the number of R-points in the volume Δ_T is given by

$$n = a_0 \mu, \qquad \ldots \quad (29)$$

and the number of cells by

$$a = a_0 (1 + \mu). \qquad \ldots \quad (30)$$

Therefore from (29) and (30) we get

$$\frac{n}{a} = \frac{\mu}{1 + \mu} \quad \text{whence} \quad \mu = \frac{n}{a - n}. \qquad \ldots \quad (31)$$

As in article 5, and with a better justification in this case,—since μ is quite small—we can take as a first approximation, $\mu = n/a$ and we obtain $\mu = e^{(\psi - U)/\Theta}$.

For the next approximation we take μ to given by (31); that is, we put $\frac{n}{a} = \frac{\mu}{1 - \mu}$, μ having the above value. Therefore we get

$$n_r = \frac{a_r}{e^{-(\psi - U)/\Theta} + 1} \qquad \ldots \quad (32)$$

whence

$$\nu = \Sigma n_r = \frac{c}{e^{-(\psi - U)/\Theta} + 1} \qquad \ldots \quad (33)$$

which again leads to

$$N = \frac{2\pi V (2m)^{\frac{3}{2}}}{h^3} \int_0^\infty \frac{U^{\frac{1}{2}} . dU}{e^{-(\psi - U)/\Theta} + 1} \qquad \ldots \quad (34)$$

instead of (28). This is the known expression for Fermi-Dirac Statistics.

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* In other words, the existence of cells containing more than one R-point is highly improbable. Therefore, we cannot justly take into account (or calculation) the number of cells with two or more R-points. Looked from this point of view Bose-Einstein's statistics is invalid in such cases.

9. It is evident from what has been said before, that the above statistics is based on the ratio n/a being small. This can always be ensured, for, a large number of similar systems but not an indefinitely large number is required for the calculation of probable or 'average' property of some sample system. The size of a cell being very small, their number within a volume Δ_T would be quite large so that we can take a sufficiently large number of systems and yet keep this number less than a.

In dealing with atoms or molecules, the size h^3 for a cell in a six-dimensional phase-space suffices for most, if not all problems we have to consider. It seems to us, therefore, that what we have expressly stated above is tacitly assumed by previous statisticians. For example, Fermi and Dirac have assumed that a cell may contain one R-point or none and that with equal probability, so that some cells remain empty whilst others have only one. This can only happen if the number of R-points be less than the number of cells ; at the most they can be equal. In some particular cases of so-called degeneracy the aid of a further application of Pauli's exclusion principle is sought to enable a cell to contain more than one R-point. Evidently this is directly opposed to the original assumption of Fermi and Dirac; in other words, their statistics fail in such cases.

On the other hand, our assumptions in Art. 8 lead us to this : that there are possibly some cells containing more than one R-point, but their number is so small that they can be neglected from our calculations—in most cases,—without materially affecting the probability result. After all, statistics can give results which are true up to a certain degree of approximation. Besides, the size of a cell not being definitely fixed it can be suitably reduced whenever necessary, so that an original cell can include a number of the new and smaller cells either wholly or in part. Thus it becomes possible for the original cell to contain two or even more R-points with greater probability than before, although the new cells are either empty or have only one R-point.

10. Let us see how the formula (32) is modified when we change our cell from size h^3 to h^3/λ, λ being not less than unity. Let n

denote the number of R-points within the volume $\Delta\tau$, a the number of original cells (of size h^3) and a' that of the new cells inside this volume. The size of the new cell should be so chosen that n/a' is sufficiently small. Then

$$a' \Big/ a = \frac{\lambda \,.\, \Delta\tau}{h^3} \Big/ \frac{\Delta\tau}{h^3} = \lambda, \quad \text{or} \quad a' = a\lambda \qquad \ldots \quad (35)$$

Now,

$$\frac{n}{a'} = \frac{n}{a\lambda} = \frac{\mathrm{N}\,.\,\varepsilon}{\mathrm{A}\,.\,\lambda}\, e^{(\psi-\mathrm{U})/\Theta}$$

from (19), or

$$\frac{n}{a'} = \frac{1}{\lambda}\, e^{(\psi-\mathrm{U})/\Theta} \qquad \ldots \quad (36)$$

by using the same value of the constant A as given in (20). If we put

$$e^{(\psi'-\mathrm{U})/\Theta} = \frac{1}{\lambda}\, e^{(\psi-\mathrm{U})/\Theta}$$

this reduces to

$$\frac{n}{a'} = e^{(\psi'-\mathrm{U})/\Theta}. \qquad \ldots \quad (37)$$

The same form could also have been obtained by making $\mathrm{A} = \frac{\mathrm{N}\varepsilon}{\lambda}$ instead of $\mathrm{N}\varepsilon$; the quantity ψ should then be different from its previous value, as is evident from (37). In this case, since n/a' is small, the equation (32) is replaced by

$$n_r = \frac{a'_r}{e^{-(\psi'-\mathrm{U})/\Theta} + 1} = \frac{a_r}{e^{-(\psi-\mathrm{U})/\Theta} + \lambda^{-1}} \qquad \ldots \quad (38)$$

This then should be the proper form for Fermi-Dirac's statistics.* Although, as has been mentioned before, λ would be unity for most of the actual cases, it may be necessary to give other values to this quantity in some special cases. As we cannot change the size of the cell arbitrarily,—this being determined by the conditions of the problem in hand, for, otherwise the results given in (13) and preceding considera-

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* **The corresponding modification for Bose-Einstein's statistics can be easily deduced if desired.**

tions, may not be valid.—it is clear that λ should, in general, be a finite number; when the distribution is continuous it becomes then necessary to give an indefinitely large value to λ. The second term in the denominator of (38) becomes negligible and we get the usual formula for continuous distribution.

11. As the ratio of the number of R-points to the number of the cells (in a small definite region of the phase-space) is always assumed to be small in our statistics, it follows that $e^{-(\psi-U)/\Theta}$, of equation (19), must be small; that is, $e^{-(\psi'-U)/\Theta}$ must always be large. The above-mentioned ratio can invariably be made small by reducing the size of a cell in a suitable manner; $e^{-(\psi'-U)/\Theta}$, and not $e^{-(\psi-U)/\Theta}$, then being large. By this means we always preclude the possibility of the exponential term in the usual Fermi-Dirac's statistics from being small in comparison with unity, even in degenerate cases.

Suppose, for example, n/a is not small, a denoting the number of cells of size h^3. Let $\frac{n}{a} = e^{-(\psi-U)/\Theta} = q$, say, a finite number (not quite small). We reduce the extension of a cell as in Art. 10, so that $\frac{q}{\lambda}$ is sufficiently small. From (38) we have

$$n_r = \frac{a_r}{q^{-1} + \lambda^{-1}} = a_r \frac{q \cdot \lambda}{q + \lambda}$$

When λ is large compared with q, the fraction can be seen to be nearly equal to q; that is, $n_r = a_r q$, an expected result.

Further, in computing average values we have restricted the available phase-space to a considerable extent, by the device of dividing the whole phase-space into very small cells. For, only those cells count which contain R-points in calculating averages and they form a very small part of the whole phase-space (see Art. 8). Moreover, the reduction of the size of a cell further narrows down the available phase-space.

We cannot close without discussing how our statistics behave at Absolute Zero. According to the usual Fermi-Dirac's statistics there is

only one cell of extension h^3 and it can contain only one R-point, the remaining R-points falling into cells of as low an energy as possible. Our statistics will require in this case,—as n/a is clearly not sufficiently small,—smaller cells. Therefore we shall have a large number of the smaller cells even within an original cell of volume h^3; and they will mostly be empty or have only one R-point. The one smaller cell corresponding to Absolute Zero will contain one R-point or at the most a finite number of them. The remaining and by far the larger part of the R-points shall occupy cells near about the above particular cell, that is, cells of very low energy. Hence the energy of our system will remain finite even at Absolute Zero. It is highly unlikely that any of the original cells will remain here empty; each of them will include a number of R-points which may be one, two, or even more depending on the circumstances of the case under consideration.

SURFACES WHOSE ASYMPTOTIC LINES CONSTITUTE A SET OF PARALLELS

BY V. RANGACHARIAR,
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1. Introduction.—In a recent paper* by the present writer, it was shown that the Mainardi-Coddazi relations may be written:

$$\frac{\partial}{\partial s}\left(\psi^2\sqrt{-K}\right) = \psi^2 J \bar{k} \qquad \ldots \quad (1)$$

and

$$\frac{\partial}{\partial v}\left(E\sqrt{-K}\right) = \sqrt{E}\,\frac{\partial}{\partial u}\left(J\psi\right) \qquad \ldots \quad (2)$$

v = constant being a system of asymptotic lines, $\bar{k}$ the curvature of the asymptotic line, and ψ the distance function for the family. If the asymptotic lines be also parallels $\psi = 1$ by a suitable choice of parameter, and (1) and (2) may be written:

$$\frac{\partial}{\partial u}\left(\sqrt{-K}\right) = JH\bar{k}\,; \qquad \ldots \quad (3)$$

and

$$\frac{\partial}{\partial v}\left(H^2\sqrt{-K}\right) = H\frac{\partial}{\partial u}J \qquad \ldots \quad (4)$$

It was also pointed out in the same paper that the J and K eliminant of (3) and (4) and the Gaussian characteristic equation

$$-KH = H_{22} \qquad \ldots \quad (5)$$

gives a certain partial differential equation for H which seems to be unsolvable.

The object of the present paper is to find some particular types of surfaces having a set of parallel asymptotic lines. The surfaces treated are surfaces of constant second and first curvatures, surfaces of revolution, ruled surfaces, and surfaces whose asymptotic lines cut at a constant angle.

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* *Bull. Cal. Math. Soc.*, Vol. XXIII, No. 2.

2. Surfaces with constant second Curvature.—If K be constant the equation (3) gives $J\bar{k} = 0$, *i.e.*, the surface is either ruled or minimal, the asymptotic lines v = constant being the generators. If the surface be ruled, since the generators are both geodesics and parallels, the surface must be developable, possessing as it does a set of parallel geodesics. If however the surface be minimal, since K is constant we obtain from (4) that

$$\frac{\partial}{\partial v}(H^2) = 0$$

i.e., H is a function of u also and the surface is still developable. Hence the theorem that "*A surface of constant second curvature and possessing a set of parallel asymptotic lines is developable.* ... (6)

The second curvature if instead of being an absolute constant be constant along the set of parallel asymptotic lines, *i.e.*, a function of v alone, the surface is still ruled or minimal from equation (3). If the surface be ruled, the asymptotic lines are again both geodesics and parallels and the surface is still developable. If however the surface be minimal, the orthogonal trajectories of the parallel asymptotic lines are geodesics and also asymptotic lines. Hence the surface is ruled and is therefore a right helicoid because the right helicoid is the only real ruled minimal surface. The same result is obtained by eliminating K between

$$-K = \frac{H_{22}}{H}$$

and $H^2\sqrt{-K} = 1$ by a suitable choice of the parameter u, from (4) and solving the resulting equation in H.

Hence, *the only surfaces having a constant second curvature along a set of parallel asymptotic lines are either developables or right helicoids.* ... (7)

3. Surfaces with constant first curvature—If J be constant, the equation (4) gives $H^2\sqrt{-K}$ is a function of u alone and therefore equal to unity by a suitable choice of the parameter u. Hence

$$-K = \frac{1}{H^4}. \quad \ldots \quad (8)$$

Eliminating K between (8) and the Gaussian characteristic equation we obtain

$$H_{22} = H^{-3}. \quad \ldots \quad (9)$$

Substituting $\sqrt{-K} = \frac{1}{H^2}$

in (3) we get
$$\frac{\partial}{\partial u}(H^{-2}) = -JH_2$$

i.e.,
$$H_1 = \frac{J}{2} H^3 H_2 \quad \ldots \quad (10)$$

Integrating (10)
$$cH^3 u + v = \phi(H) \quad \ldots \quad (11)$$

where
$$J = 2c.$$

Differentiating (11) twice in succession with respect to v, we have
$$3ucH^2H_2 + 1 = \phi'(H)H_2$$

and
$$6uc\,HH_2^2 + 3ucH^2H_{22} = \phi''(H)H_2^2 + \phi'(H)H_{22}$$

giving
$$H_{22} = \frac{6ucH - \phi''}{(\phi' - 3ucH^2)^3}$$

Substitution for H_{22} in (9) gives
$$\frac{\phi'' - 6ucH}{(\phi' - 3ucH^2)^3} = -\frac{1}{H^3} \quad \ldots \quad (12)$$

Integration with respect to H gives
$$\phi - ucH^3 = A\sqrt{H^2 - A^2} + B \quad \ldots \quad (13)$$

Hence (from 11)

we obtain
$$H^2 = \left(\frac{v - B}{A}\right)^2 + A^2.$$

H is thus a function of v only and so also is K. Hence from (3) the surface is minimal.

"*The only surface with constant first curvature and possessing a set of parallel asymptotic lines is minimal and hence a right helicoid*" ... (14)

4. Surface of Revolution—For a surface of revolution we have

$$x = u\cos v, \quad y = u\sin v, \quad z = f(u),$$
$$E = 1 + f'^2, \quad F = 0, \quad G = u^2,$$
$$HL = uf'', \; M = 0, \; HN = u^2 f'.$$

The asymptotic lines are given by

$$\frac{dv}{du} = \pm \sqrt{\frac{-f''}{uf'}}$$

Let us assume that

$$\frac{dv}{du} = +\sqrt{\frac{-f''}{uf'}}$$

constitute a set of parallels. Then their orthogonal trajectories are geodesics. The orthogonal trajectory of

$$\frac{dv}{du} = \sqrt{\frac{-f''}{uf'}}$$

is given by

$$(1 + f'^2) \sqrt{-f'^1}\, du + u^2 \sqrt{-f''}\, dv = 0$$

i.e.,

$$\frac{dv}{du} = -\frac{1 + f'^2}{u^2} \sqrt{\frac{uf'}{-f''}} \qquad \ldots \quad (15)$$

Now the first integral for geodesics on a surface of revolution is given by

$$u^4\, dv^2 = c^2 \{ (1 + f'^2)\, du^2 + u^2\, dv^2 \} \qquad \ldots \quad (16)$$

It is clear that if

$$\frac{dv}{du} = -\sqrt{\frac{uf'}{-f''}}\ \frac{1 + f'^2}{u^2}$$

satisfies (16) then

$$\frac{dv}{du} = +\sqrt{\frac{uf'}{-f''}}\ \frac{1 + f'^2}{u^2}$$

also satisfies it.

Hence *if one system of asymptotic lines on a surface of revolution be a set of parallels, so also is the other.*

From (16) and

$$\frac{dv}{du} = \pm \frac{1 + f'^2}{u^2} \sqrt{\frac{uf'}{-f''}}$$

we get

$$(u^2 - c^2)(1 + f'^2) f' + c^2 uf'' = 0$$

whence integrating we obtain

$$f(u) = \int \frac{u e^{a - a^2/2c^2}}{\sqrt{1 - u^2 e^{2a - u^2/c^2}}}\, du.$$

The only surface of revolution having a set of parallel asymptotic lines is one formed by the revolution of the curve

$$y = \int \frac{xe^{a-x^2/2c^2}}{\sqrt{1-x^2e^{2a-x^2/c^2}}}\,dx$$

about the y axis. ... (18)

5. Ruled surface with its curved asymptotic lines as a set of parallels.

Taking the generators as $v =$ constant and their orthogonal trajectories as $u =$ constant we get $ds^2 = du^2 + \psi^2 dv^2$. In that case

$$\psi^2 = (u-\alpha)^2 + \beta^2;$$

$$J = \frac{1}{\psi}\left\{\mu - \frac{\beta'(u-\alpha) + \alpha'\beta}{\psi^2}\right\};$$

and $K = \dfrac{-\beta^2}{\psi^4}$ where α. β, μ are functions of v alone and accents dashes denote differentiation with regard to v.

The curved asymptotic line is given by

$$2\sqrt{-K}\,du + J\psi\,dv = 0,$$

i.e.,

$$2\beta\,du + J\psi^3\,dv = 0.$$

Its orthogonal trajectory which is

$$J\psi\,du - 2\beta\,dv = 0$$

is a geodesic and therefore satisfies

$$\psi^2\psi_1\left(\frac{dv}{du}\right)^2 + \psi_2\left(\frac{dv}{du}\right) + 2\psi_1 = 0 \qquad \text{... (19)}$$

the differential equation of the geodesics.

From (19) and (20)

$$\psi^2\psi\psi_1\left(\frac{J\psi}{2\beta}\right)^2 + \psi\psi_2\frac{J\psi}{2\beta} + 2\psi\psi_1$$

must identically vanish. Hence

$$(u-\alpha)\ \{(u-\alpha)^2+\beta^2\} \times \left\{\mu - \frac{\beta'(u-\alpha)+\alpha'\beta}{(u-\alpha)^2+\beta^2}\right\}^2$$

$$+ 2\beta \left\{\mu - \frac{\beta'(u-\alpha)+\alpha'\beta}{(u-\alpha)^2+\beta^2}\right\}\{\beta\beta'-\alpha'(u-\alpha)\}$$

$$+ 8\beta^2(u-\alpha) \equiv 0.$$

Simplifying and equating to zero the co-efficients of different powers of $u-\alpha$ we get

$$\left.\begin{aligned} \mu &= 0, \\ \beta'^2 + 8\beta^2 &= 0, \\ \alpha'\beta\beta' &= 0, \\ 3\alpha^2\beta^2 + 8\beta^4 - 2\beta^2\beta'^2 + 2\alpha'^2\beta^2 &= 0, \\ \beta^3\alpha'\beta' &= 0. \end{aligned}\right\} \quad \ldots \quad (20)$$

The only possible solution of these equations is $\beta = 0$. Then since

$$K = \frac{-\beta^2}{\psi^4} \quad \text{and} \quad J = \frac{1}{\psi}\left\{\mu - \frac{\beta'(u-\alpha)+\alpha'\beta}{\psi^2}\right\}$$

we have $K = 0$ and $J = 0$ because μ and β both vanish.

Hence

the only ruled surfaces whose curved asymptotic lines constitute a set of parallels are planes. ... (21)

6. Surfaces whose asymptotic lines cut at a constant angle.— The non-parallel asymptotic lines are given by $2\,M du + N dv = 0$, or ince $-K = M^2/H^2$ and $J = N$, the same may be written

$$2H\sqrt{-K}\,du + J dv = 0.$$

This cuts the curve $v =$ constant—the parallel asymptotic line—at an angle given by

$$\tan\psi = \frac{H\sqrt{-4H^2K}}{H^2J} = \pm\frac{2\sqrt{-K}}{J}.$$

Hence if the asymptotic lines cut at a constant angle,

$$\sqrt{-K} = aJ.$$

Substitution in (3) and (4) gives

$$a\frac{J_1}{J} = -H_2$$

and $$a\frac{\partial}{\partial v}(H^2 J) = H J_1$$

Hence $$\frac{\partial}{\partial v}(H^2 J) = \frac{H}{a} J_1 = -JHH_2$$

$$= -H^2 J \frac{H_2}{H}$$

$\therefore$ $$\frac{\partial}{\partial v}(\log H^2 J) + \frac{\partial}{\partial v}\log H = 0$$

Hence $H^3 J$ is a function of u alone and therefore unity by a suitable choice of the parameter u.

Hence $$-H_2 = a\frac{\partial}{\partial v}(\log J)$$

$$= -3a\frac{\partial}{\partial v}\log H$$

or $$3a\, H_1 - HH_2 = 0. \qquad \ldots \quad (22)$$

Hence $$H = \phi\,(u + 3\,av).$$

Now the Gaussian characteristic equation gives

$$a^2 J^2 = \frac{H_{22}}{H}$$

i.e., $$a^2 = H_{22}\, H^5 = 9\,a^2\,\phi''\phi^5$$

$\therefore$ ϕ satisfies the differential equation

$$9\,\phi''\,\phi^5 = 1.$$

Integrating and separating the variables we get

$$\frac{A^2\,\phi^2\,d\phi}{\sqrt{\phi^4 - A^4}} = \frac{dx}{3\sqrt{2}} \text{ where } x = u + 3av.$$

Hence H is an elliptic function of $u + 3av$.

The only surfaces whose asymptotic lines cut at a constant angle and which possesses a set of parallel asymptotic lines are given by

$$ds^2 = H^2\,dv^2 + dv^2$$

where H is a certain Elliptic function of $u + 3av$, the parallel asymptotic lines and their orthogonal trajectories being taken as the parametric curves. ... (23)

THE CURVE OF POPULATION.

BY M. V. JAMBUNATHAN, M.A., B Sc.,

Mysore University.

[*Syncpsis.*—In this paper an attempt is made to derive the equation to the curve of population of any country from an *a priori* consideration of the nature and limitations of the *rate* of growth of population.

The equation obtained is

$$y = \frac{M}{1 + e^{a_0 + a_1 x}},$$

y being the population at time x, M the maximum or the '*optimum* population, and a_0, a_1 constants. A slightly extended form of the equation is

$$y = \frac{M}{1 + e^{a_0 + a_1 x + a_2 x^2 + a_3 x^3}}.$$

The methods of calculating M and the other constants, *viz.*, a_0, a_1, ..., are also discussed here.]

The rate of growth of population (with respect to time) of any country is a function of the population itself, the time, the stage of its civilization, the material resources of the country, etc.; and this can be expressed symbolically in the form

$$\frac{dy}{dx} = f(x, y, z, u \ldots) \qquad \ldots \quad (1)$$

where x is the time measured from any epoch, y the population, and z, u, ... other factors.

Other things remaining the same, it is obvious that the rate of growth of population increases with y, or

$$\frac{\partial}{\partial y}\left(\frac{dy}{dx}\right) = \frac{\partial f}{\partial y} = f_1(x, z, u, \ldots) \qquad \ldots \quad (2)$$

From *a priori* considerations it is evident that there is an upper limit to the human population, and as the population approaches this limit, the pressure on the means of subsistence is increased, and this acts as a check to the normal growth of population, *i.e.*, the growth is slowed. If M be the upper limit to the population, and if $M-y$ be denoted by z, then the smaller the value of z, the greater is the check on the growth of population, *i.e.*, the smaller the rate of growth. That is, we may assume that

$$\frac{\partial}{\partial z}\left(\frac{dy}{dx}\right) = \frac{\partial f}{\partial z} = f_2(x, y, u, \ldots) \qquad \ldots \quad (3)$$

From equations (2) and (3) we infer that equation (1) is of the type

$$\frac{dy}{dx} = y \,.\, z \,.\, \phi(x, u, \ldots) \qquad \ldots \quad (4)$$

where $\phi(x, u, \ldots)$ is a function of (1) the time, (2) the area of the country whose population growth we are considering, (3) the stage of its civilization, etc.. This is the general equation of the growth of population. For practical purposes we will have to take $\phi(x, u \ldots)$ as a function of only x, *i.e.*, of time, for it is not possible to assess these others quantitatively. Further, if the period we are considering be comparatively short, say, half a century or so, no great change in these other factors would have taken place and hence we may ignore the influence of these factors on the growth of population. Hence equation (4) must, for all practical purposes, be taken as

$$\frac{dy}{dx} = y \,.\, z \,.\, \phi(x) \qquad \ldots \quad (5)$$

or

$$\frac{1}{y.(M-y)} \cdot \frac{dy}{dx} = \phi(x). \qquad \ldots \quad (6)$$

Integrating,

$$\log \frac{y}{M-y} = \int \phi(x)\, dx$$

or

$$\frac{y}{M-y} = e^{\int \phi(x) dx}$$

from which

$$y = \frac{M}{1 + e^{-\int \phi(x)\, dx}}.$$

If we further assume that $\phi(x)$ can be expressed as a power series, then

$$y = \frac{M}{1 + e^{a_0 + a_1 x + a_2 x^2 + \dots a_n x^n}}. \quad \dots \quad (7)$$

Since the population never actually reaches M, but approaches M more and more closely with the increase of time, the line $y = M$ will be asymptotic to the curve. Also since we cannot have negative populations, the line $y = 0$ will be the lower asymptote touching the curve at $-\infty$, Further we notice that if $y = M$ is to be asymptotic to the curve, touching it at $+\infty$, then a_n is negative. Also $y = 0$ should be asymptotic to the negative branch of the curve, and hence n is odd.

Bearing in mind these restrictions, we may write as a first approximation that

$$y = \frac{M}{1 + e^{a_0 + a_1 x}} \quad \dots \quad (8)$$

from which we have

$$a_0 + a_1 x = \log \frac{M - y}{y}.$$

If we choose three points on the graph which are uniformly spaced along the X-axis, *viz.*, $(0, y_0)$, (x_1, y_1), $(2x_1, y_2)$, we have

$$a_0 = \log \frac{M - y_0}{y_0}$$

$$a_0 + a_1 x_1 = \log \frac{M - y_1}{y_1}$$

$$a_0 + 2a_1 x_1 = \log \frac{M - y_2}{y_2}.$$

Eliminating a_0 and a_1, we get

$$\left(\frac{M - y_1}{y_1}\right)^2 = \left(\frac{M - y_0}{y_0}\right)\left(\frac{M - y_2}{y_2}\right)$$

$$M = \frac{2y_0 y_1 y_2 - y_1^2 (y_0 + y_2)}{y_0 y_2 - y_1^2},$$

rejecting the other value, *viz.*, $M = 0$.

Thus $a_0 = \log \frac{M - y_0}{y_0}$, and $a_1 = \frac{1}{x_1} \cdot \log \frac{y_0 (M - y_1)}{y_1 (M - y_0)}$.

When we have more *observed* points than three, (say r) we may get a second approximation if we so alter the constants as to get a curve of best fit.

Let α be an approximate value of a_1 [we may take α as $\frac{2y_0 y_1 y_2 - y_1^2 (y_0 + y_2)}{y_0 y_2 - y_1^2}$], and let the correction to be applied to it be h, so that

$$a_1 = \alpha + h.$$

Equation (8) may be written

$$y = \frac{b}{c + e^{a_1 x}}, \qquad \ldots \quad (9)$$

where $c = e^{-a_0}$, and $b = M e^{-a_0}$.

$$\therefore \quad y = \frac{b}{c + e^{(\alpha + h)x}}$$

$$= \frac{b}{c + e^{\alpha x}\left(1 + hx + \frac{h^2 x^2}{2!} \ldots\right)}$$

$$= \frac{b}{c + e^{\alpha x} + hx \cdot e^{\alpha x}}$$

neglecting the second and higher powers of h;

$$i.e \qquad yc + ye^{\alpha x} + hxy\, e^{\alpha x} - b = 0. \qquad \ldots \quad (10)$$

By the method of least squares we derive the normal equations

$$c \Sigma y + \Sigma ye^{\alpha x} + h \Sigma xye^{\alpha x} - b \cdot r = 0$$

$$c \Sigma y^2 + \Sigma y^2 e^{\alpha x} + h \Sigma xy^2 e^{\alpha x} - b \Sigma y = 0$$

$$c \Sigma xy^2 e^{\alpha x} + \Sigma xy^2 e^{2\alpha x} + h \Sigma x^2 y^2 e^{2\alpha x} - b \Sigma xye^{\alpha x} = 0.$$

Thus the values of b, c, and h can be found out and hence of a_1, and this value is a better approximation than that obtained from equation (8).

If we are prepared to introduce more constants than three, we will have to take the equation

$$y = \frac{M}{1 + e^{a_0 + a_1 x + a_2 x^2 + a_3 x^3}} \quad \dots \quad (11)$$

which can be easily reduced to the form

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 = \log \frac{M - y}{y}.$$

If we choose five points whose ordinates are equidistant [we may take these points as $(0, y_0)$, (x_1, y_1) $(2x_1, y_2)$ $(3x_1, y_3)$ and $(4x_1, y_4)$], we have

$$\begin{aligned}
a_0 & & & & = \log \frac{M - y_0}{y_0} \\
a_0 + a_1 x_1 &+ a_2 x_1^2 &+ a_3 x_1^3 & & = \log \frac{M - y_1}{y_1} \\
a_0 + 2a_1 x_1 &+ 4a_2 x_1^2 &+ 8a_3 x_1^3 & & = \log \frac{M - y_2}{y_2} \\
a_0 + 3a_1 x_1 &+ 9a_2 x_1^2 &+ 27a_3 x_1^3 & & = \log \frac{M - y_3}{y_3} \\
a_0 + 4a_1 x_1 &+ 16a_2 x_1^2 &+ 64a_3 x_1^3 & & = \log \frac{M - y_4}{y_4}
\end{aligned}$$

Eliminating a_0, a_1, a_2, a_3, we get

$$\frac{M - y_0}{y_0} \left(\frac{M - y_2}{y_2}\right)^6 \cdot \frac{M - y_4}{y_4} = \left(\frac{M - y_1}{y_1}\right)^4 \left(\frac{M - y_3}{y_3}\right)^4 \quad \dots \quad (12)$$

from which M can be calculated. It is then easy to get the values of a_0, a_1, a_2, and a_3.

Now equation (12) has to be solved by Horner's method of approximation, bearing in mind that $M > y_4$.

The theoretical investigation into the nature of the equation and its roots is being pursued and the results of the investigation will be given in a subsequent paper.

DETERMINANTS AND THE ROOTS OF AN EQUATION

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Two theorems have been proved which allow the number of distinct roots, and of distinct real roots, of an equation with real coefficients to be expressed in terms of certain properties of a determinant. The better known of these is Borchardt's result :

Given

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots \dots + a_n = 0, \qquad \dots \quad (1)$$

with real coefficients, form the determinant

$$\begin{vmatrix} n & s_1 & s_2 & \dots \dots & s_{n-1} \\ s_1 & s_2 & s_3 & \dots \dots & s_n \\ s_2 & s_3 & s_4 & \dots \dots & s_{n+1} \\ \dots & \dots & \dots & \dots \dots & \dots \\ \dots & \dots & \dots & \dots \dots & \dots \\ s_{n-1} & s_n & s_{n+1} & & s_{2n-2} \end{vmatrix} \qquad \dots \quad (2)$$

where s_i means the sum of the ith powers of the roots of (1). Then the number of distinct roots of (1) is equal to the rank of (2)*, while the number of distinct real roots is equal to the signature of (2).†

The second part of this statement may be given an equivalent form :

Set up the series of principal minors

$$n, \quad \begin{vmatrix} n & s_1 \\ s_1 & s_2 \end{vmatrix}, \quad \begin{vmatrix} n & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} \quad \dots, \quad \begin{vmatrix} n & s_1 & s_2 & \dots & s_{r-1} \\ s_1 & s_2 & s_3 & \dots & s_r \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ s_{r-1} & s_r & s_{r+1} & \dots & s_{2r-2} \end{vmatrix} \quad \dots \quad (3)$$

---

* (2) itself is, as is well-known, the discriminant of (1).

† or more exactly, of the quadratic form whose discriminant is (2).

where r denotes the rank of (2).* Determine the sign of each of these principal minors. If the resulting sequence of signs has k variations, then (1) has $r - 2k$ distinct real roots.

This criterion may fail in case two consecutive terms of (3) vanish. However, by a familiar theorem on symmetric determinants, it is then possible to rearrange rows and corresponding columns of (2), keeping n in its corner position, so that the series of principal minors similar to (3) do not have two consecutive zero terms. This series may then be used in place of (3).

The other theorem of the same type is given by Weber in his *Algebra*,† and also by Fricke.‡ It is developed as an application of a form of the Tschirnhaus transformation invented by Hermite and used by a number of writers. This transformation on the roots of (1) is

$$y = t_0 . F_0(x) + t_1 . F_1(x) + \dots \dots + t_{n-2} F_{n-2}(x), \quad \dots \quad (4)$$

where the t_i are parameters which can be chosen at will, and where

$$F_0(x) = x + \frac{1}{n} a_1.$$

$$F_1(x) = x^2 + a_1 x + \frac{2}{n} a_2,$$

$$\dots \qquad \dots$$

$$\dots \qquad \dots$$

$$F_{n-2}(x) = x^{n-1} + a_1 x^{n-2} + \dots \dots + a_{n-2}\, x + \frac{n-1}{n} a_{n-1}.$$

In the transformed equation the coefficient of y^{n-1} is obviously zero, while the other coefficient possess a certain invariantive property which we do not need here. We require only the coefficient of y^{n-2}, or, more exactly, -2 times this coefficient, which is obtained by squaring (4) and summing over the roots of (1). The result is ordinarily denoted by Σy^2; it is a quadratic form in the t_i. That is, we have

$$\Sigma y^2 = \sum_{i, K} B_{ik}\, t_i\, t_k \qquad (i, k = 0, 1, \dots \dots n-2),$$

* Series (3) thus has no further non-vanishing terms.

† Weber does not credit the result to an earlier writer, but it actually dates back to Sylvester.

‡ *Lehrbuch der Algebra*, Vol. 1, 1924, p. 189.

where it is easily seen, from (4) ,that

$$B_{ik} = \Sigma F_i(x) F_k(x),$$

this last summation being again over the roots of (1).

The discriminant of the quadratic form Σy^2 is

$$\begin{vmatrix} B_{00} & B_{01} & \dots\dots & B_{0,n-2} \\ B_{10} & B_{11} & \dots\dots & B_{1,n-2} \\ \dots & \dots & \dots\dots & \dots \\ \dots & \dots & \dots\dots & \dots \\ B_{n-2,0} & B_{n-2,1} & \dots\dots & B_{n-2,n-2} \end{vmatrix} \quad \dots \quad (5)$$

and the theorem we are interested in is as follows :

Let the rank of (5) be denoted by R, which is at most $n-1$. Set up the series

$$M_0 = 1,\ M_1 = B_{00},\ M_2 = \begin{vmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{vmatrix},\ M_3,\ \dots\ \dots,\ M_R, \quad \dots \quad (6)$$

the terms after M_0 being successive principal minors of (5).* Determine the sign of each term of (6), and let k be the number of variations of sign. Then (1) has $R+1$ distincts roots, of which $2k$ are complex.

Now (2) is a determinant of order, n, while (5) is of order $n-1$. and the elements of the latter do not bear any obvious relation to those of (2).† It thus becomes a matter of some interest to study the two, to decide whether either may give simpler criteria than the other. Such a situation might well arise ; it is well-known that, for the quartic $x^4 + px^2 + qx + r = 0$, the nature of the roots can be ascertained from the signs of the discriminant, the coefficient p, and either $p^2 - 4r$ or $-2p^3 + 8pr - 9q^2$.

---

* The statement made in the paragraph after (3) again applies here, in case two consecutive terms of (6) vanish. The last term M_R cannot vanish though we have as yet no proof of this fact ; if it could vanish, we should simply need another rearrangement of rows and columns of (5), for there is at least one non-vanishing R-rowed minor.

† Fricke does prove that n times (5) is equal to (2).

To compare (2) and (5) conveniently, we obtain another determinant which can be used in place of them, and which is related in a simple way to each of them. This determinant arises from a consideration of the standard Tschirnhaus transformation

$$y = c_{n-1}\left(X^{n-1} - \frac{1}{n}s_{n-1}\right) + c_{n-2}\left(X^{n-2} - \frac{1}{n}s_{n-2}\right) + \dots + c_1\left(X - \frac{1}{n}S_1\right),$$

where the constant term on the right-hand side is determined so that $\Sigma y = 0$. We can form Σy^2 easily; it is a quadratic form in the c's with discriminant

$$\begin{vmatrix} s_2 - \frac{1}{n}s_1^2 & s_3 - \frac{1}{n}s_1 s_2 & \dots & s_n - \frac{1}{n}s_1 s_{n-1} \\ s_3 - \frac{1}{n}s_2 s_1 & s_4 - \frac{1}{n}s_2^2 & \dots & s_{n+1} - \frac{1}{n}s_2\, s_{n-1} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ s_n - \frac{1}{n}s_{n-1}s_1 & s_{n+1} - \frac{1}{n}s_{n-1}s_2 & \dots & s_{-2} - \frac{1}{n}s^2_{n-1} \end{vmatrix} \quad \dots \quad (7)$$

And by a proof almost the same as that given by Fricke we can show that (7) has the same property as (5), that is that the series

$$N_0 = 1,\ N_1 = s_2 - \frac{s_1^2}{n}\ N_2 = \begin{vmatrix} s_2 - \frac{s_1^2}{n} & s_3 - \frac{s_1 s_2}{n} \\ s_3 - \frac{s_1 s_2}{n} & s_4 - \frac{s_2^2}{n} \end{vmatrix}, \dots \dots, N_R. \quad \dots \quad (8)$$

where R is now the rank of (7), can be used in place of (6), and also, of course, in place of (3).

We next have to study more carefully the relations between (8) and (3), and (8) and (6). First, each term of (8) is exactly the corresponding term of (3) divided by n. For, if we divide the first row (or column) of (2) by n and apply Chio's method of reducing a determinant of order n to one of order $n-1$, we get (7). The same reduction changes each minor in (3) into n times the corresponding term in (8), and incidentally shows that the two series actually have the same number of terms, as they of course must have. Stated a little differently, series (8) is really simply series (3) with part of the evaluation of the terms carried out. It is thus a little easier to use than (3); this is perhaps especially so if we have $s_1 = 0$.

The comparison of (8) and (6) must be carried out a little differently. The quadratic forms leading to (5) and (7) are forms in $t_0, t_1, \dots \dots, t_{n-2}$, and in $c_1\ c_2, \dots \dots\ c_{n-1}$, respectively. An examination of the equations preceding (5) and (7) shows, after a little computation, that the t_i and c_i are connected by the following relations:

$$
\begin{aligned}
c_1 &= t_0 + a_1 t_1 + a_2 t_2 + a_3 t_3 + \dots \dots + a_{n-2}\, t_{n-2} \\
c_2 &= \quad t_1 + a_1 t_2 + a_2 t_3 + \dots \dots + a_{n-3}\, t_{n-2} \\
c_3 &= \quad\quad t_2 + a_1 t_3 + \dots \dots + a_{n-4}\, t_{n-2} \\
&\dots \quad \dots \quad \dots \quad \dots \\
&\dots \quad \dots \quad \dots \quad \dots \\
c_{n-1} &= \quad\quad\quad\quad t_{n-2}.
\end{aligned}
\qquad \dots \quad (9)
$$

Equations (9) determine a linear transformation of determinant

$$
\begin{vmatrix}
1 & a_1 & a_2 & \dots \dots & a_{n-2} \\
0 & 1 & a_1 & \dots \dots & a_{n-3} \\
0 & 0 & 1 & \dots \dots & a_{n-4} \\
\dots & \dots & \dots & \dots \dots & \dots \\
\dots & \dots & \dots & \dots \dots & \dots \\
0 & 0 & 0 & \dots \dots & 1
\end{vmatrix}
\qquad \dots \quad (10)
$$

which leads from the quadratic form in the c_i to that in the t_i. It follows, by a standard theorem on quadratic forms, that if we multiply the conjugate of (10), that is, (10) with rows and columns interchanged, by (7), and then multiply this product by (10), we get exactly (5). Since (10) has the value unity, (7) and (5) are equal. Further, an examination of (9) and (10) shows that each term of (8) must equal the corresponding term of (6).

We have thus accomplished two things; we have proved that criteria (3) and (6) are the same, and in so doing have developed another series, that is, (8), which is equivalent to the others, and which is perhaps a little easier to set up* and use than either of them.

---

* It should be mentioned that Fricke develops formulas which make (5) easier to set up than would at first appear.

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NOTES AND QUESTIONS.

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Notes and Questions.

India's Contribution to the Theory of Indeterminate Equations of the First Degree.

(*Continued from page 142*).

In solving the equation $Ax = By + 1$ where A and B are prime to each other, the younger Āryabhaṭa simplifies the elder Āryabhaṭa's rule by taking zero as the *mati* (assumed integer). Accordingly the elder Āryabhaṭa's chain

$$\begin{array}{c} Q_1 \\ Q_2 \\ Q_3 \\ \cdots \\ \cdots \\ \cdots \\ Q_{n-2} \\ Q_{n-1} \\ Q_n\, t + C \end{array}$$

becomes reduced to the following chain

$$\begin{array}{c} Q_1 \\ Q_2 \\ Q_3 \\ \cdots \\ \cdots \\ \cdots \\ Q_{n-2} \\ Q_{n-1} \\ 1 \end{array}$$

as $t = 0$ and $C = 1$.

This is younger Āryabhaṭa's chain. He does not require the quotient given by the operation of division by the unity-remainder. Accordingly

he stops the operations of mutual division when the remainder becomes unity. Like Mahāvīra he uses one quotient less than his senior namesake; but, unlike Mahāvīra he obtains the values of both the unknown quantities from his chain. Mahāvīra rejects the first quotient and uses the last, whereas the younger Āryabhaṭa rejects the last quotient and utilises the first.

Using his chain of quotients like his predecessors the younger Āryabhaṭa arrives at two numbers (say, α, β) which satisfy

either $$Ax = By + 1$$
or $$Ax = By - 1$$

according as the number of remainders up to and including the unity-remainder is even or odd, *i.e.*, according as the number of quotients in the chain is even or odd since each operation of division gives only one quotient and one remainder.[25] The reason has already been explained in connection with Māhavīra's second rule. Therefore, $C\alpha$ and $C\beta$ will be a pair of roots of the equation

$$Ax = By + C$$
or $$Ax = By - C$$

according as the number of quotients in the chain is even or odd. If $C\alpha$ and $C\beta$ be both greater than B and A respectively we may write

$$C\alpha = nB + q, \qquad C\beta = mA + p,$$

where $q < B$, $p < A$, and m, n are integers.

Then $$A(nB + q) = B(mA + p) \pm C \qquad \ldots \quad (17)$$

Subtracting lAB from both sides

$$A\{(n - l)B + q\} = B\{(m - l)A + q\} \pm C.$$

Therefore $(n - l)B + q$, $(m - l)A + p$ will be the least roots of the equation

$$Ax = By \pm C$$

if l is equal to either of m and n when they are equal or to the smaller of m and n when they are unequal.

---

[25] "एवमभीष्टविधिभवौ फलगुणकौ प्रस्फुटौ धनक्षेपे ।
समवल्ल्यां विषमयामृणसंज्ञे क्षेपके स्याताम् ॥"

Dividing both sides of (17) by AB we have

$$n + \frac{q}{B} = m + \frac{p}{A} \pm \frac{C}{AB}.$$

∴ either $n = m$,

or $n \gtrless m$ according as C is taken with positive or negative sign.

Hence Āryabhaṭa says :

" अन्यत्र प्रश्नोक्तावथ तत्सम्बन्धजे यदा लब्धी ।
न समे गुण एव तदा ग्राह्यो हेयं फलं धनक्षेपे॥
फलमृणसंज्ञे ग्राह्यं हेयो गुणको गुणात् फलोत्पत्तिम् ।
वक्ष्ये फलतोऽपि तथा सर्वत्र समां गुणोत्पत्तिम् ॥"

"If the quotient and multiplier of the equation $Ax = By \pm C$, as derived from those of the equation $Ax = By \pm 1$, do not yield equal quotients (namely, n, m) on being divided respectively by the dividend (B) and the divisor (A) for the purpose of obtaining the least roots of the equation, the quotient (m) given by the multiplier should be accepted and the quotient (n) given by the quotient should be rejected in the case of a positive *kṣepa* (*i.e.*, when C is preceded by the positive sign), and the quotient (n) given by the quotient should be accepted and that (m) given by the multiplier rejected in the case of a negative *kṣepa*. I shall now explain how the quotient can be found when the multiplier is known and the multiplier can be found when the quotient is known."

The younger Āryabhaṭa's method of deriving the roots of the equation $ax = by \pm c$ from the roots (α, β) of the reduced equation $Ax = By \pm 1$ when the third, fourth, or fifth operation has been performed is as follows :

If the third operation has been performed, $a = A$, $b = mB$, $c = mC$.

Now $$A\alpha = B\beta \pm 1$$

∴ $$Am\,C\alpha = Bm\,C\beta \pm mC$$

or $$a\,(c\alpha) = b\,(C\beta) \pm c$$

∴ $c\alpha$ and $C\beta$ are the roots of the given equation.

If the fourth operation has been performed, $b = B$, $a = mA$, $c = mC$.

Now $$A\alpha = B\beta \pm 1$$

$$\therefore \quad Am\,C\alpha = Bm\,C\beta \pm mC$$

or $$a\,(C\alpha) = b\,(c\beta) \pm c$$

$\therefore$ $C\alpha$ and $c\beta$ are the roots of the original equation.

If the fifth operation has been performed, $a = mA$, $b = nB$, $c = mnC$.

Now $$A\alpha = B\beta \pm 1$$

$$\therefore \quad Amn\alpha = Bmn\beta \pm mn$$

or $$a\,(n\alpha) = b\,(m\beta) \pm mn \qquad \ldots \quad (18)$$

[$n\alpha$ and $m\beta$ have been called the intermediate (*madhya*) quotient and multiplier respectively.]

Multiplying both sides of (18) by $\frac{c}{mn}$ we have

$$a\,\frac{n\alpha\,.\,c}{mn} = b\,\frac{m\beta\,.\,c}{mn} \pm c.$$

As c is divisible by mn, $\left(x = \frac{n\alpha\,.\,c}{mn},\ y = \frac{m\beta\,.\,c}{mn}\right)$ is an integral, solution of the equation

$$ax = by \pm c.$$

As $$mn = a\,(n\alpha) \sim b\,(m\beta),$$

$$\frac{n\alpha\,.\,c}{a\,(n\alpha) \sim b\,(m\beta)},\quad \frac{m\beta\,.\,c}{a\,(n\alpha) \sim b\,(m\beta)}$$

are a pair of integral roots of the equation.

From the least roots of the equation $Ax = By + C$ where C may be positive or negative the younger Āryabhaṭa deduces the least roots of the equation $Ax = By - C$ after the manner of Mahāvira.

The younger Āryabhaṭa considers the following particular cases of the equation $ax = by + c$:

(i) If c be positive and divisible by a, the least values of x and y are respectively c/a and 0.

(ii) If c be negative and divisible by a, ($x = c/a$, $y = 0$) will not be a solution.

(iii) If $c = 0$, $x = 0$ and $y = 0$.

From the least roots of the equation $Ax = By + C$ where A and B are prime to each other the younger Āryabhaṭa obtains the general values of x and y by adding the same multiples of B and A respectively to the least values of x and y. Thus, if α and β be the least values of x and y, the general expressions for the values of x and y are given by

$$x = \alpha + Bt, \quad y = \beta + At.$$

The younger Āryabhaṭa's rule for the solution of simultaneous indeterminate equations of the first degree in which the divisors are the same is as follows :

गुणकैक्यं संश्लिष्टे भाज्यः शेषैक्यकं भवेत् क्षेपः ।
तुल्यच्छेदे कर्म मन्दार्थं कथ्यते विततः ॥

"For the solution of simultaneous indeterminate equations in which the divisors are the same the dividend should be taken to be the sum of the multipliers and the *ksepa* to be the sum of the given *ksepas*. The process is elaborately explained for the understanding of the comparatively dull."

If the simultaneous indeterminate equations are

$$ay = bx + c \quad \ldots \quad (19)$$

and

$$az = Bx + C \quad \ldots \quad (20)$$

then

$$a(y + z) = (b + B)x + (c + C) \quad \ldots \quad (21)$$

or

$$at = (b + B)x + (c + C). \quad \ldots \quad (22)$$

The younger Āryabhaṭa says that the value of x which satisfies equations (19) and (20) satisfies equation (22) also. Bhāskara also has given the same rule in different words. Neither of them indicates that the values of x which satisfy equation (22) include other values also which do not satisfy equations (19) and (20) but satisfy equations like

$$ay = mx + n$$

and

$$az = Mx + N$$

where

$$m + M = b + B, \text{ and } n + N = c + C.$$

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It may be noted that the younger Āryabhaṭa's process of deriving the roots of the equation $Ax = By + C$ from his chain of quotients

$$\begin{array}{c} Q_1 \\ Q_2 \\ Q_3 \\ \cdots \\ \cdots \\ Q_{n-2} \\ Q_{n-1} \\ Q_n \\ 1 \end{array}$$

is exactly the same as the process of simplifying the continued fraction

$$Q_1 + \frac{1}{Q_2 +}\ \frac{1}{Q_3 +}\ \frac{1}{Q_4 +} \cdots \cdots \frac{1}{Q_{n-2} +}\ \frac{1}{Q_{n-1} +}\ \frac{1}{Q_n}.$$

If the operation of division, which gives the quotient Q_n, also gives the remainder 1, the above continued fraction is the penultimate convergent of the fraction $\frac{B}{A}$. His upper and lower *kuṭṭas* are respectively the numerator and denominator of the simplified form of this convergent. Hence the younger Āryabhaṭa's method of solution of the equation $Ax = By \pm C$ depending on the solution of the equation $Ax = By \pm 1$ is exactly the same as the modern method depending on a result in the theory of continued fractions but is independent of that result. This result, however, follows easily from the younger Āryabhaṭa's method. If p_n / q_n be the penultimate convergent of the fraction B/A, p_n, q_n are respectively the upper and lower *kuṭṭas*.

Therefore, $$Ap_n - Bq_n = \pm 1 \qquad \ldots \quad (23)$$

according as the penultimate convergent is of even or odd order. Suppose p_n / q_n and B/A to be two consecutive convergents of some fraction so that $B = p_{n+1}$ and $A = q_{n+1}$ in our ordinary notation. Therefore, from (23)

$$q_{n+1}\, p_n - p_{n+1}\, q_n = \pm 1$$

according as n is even or odd.

The preliminary operations recommended by the younger Āryabhaṭa for the simplification of the equation $ax = by + c$ may be used with advantage even now when employing the method of continued fractions.

The last Indian mathematician before the thirteenth century who treats of indeterminate equations of the first degree is Bhāskara (b. 1114 A.D.). Excepting his rule for the solution of general simultaneous indeterminate equations of the first degree he contributes practically nothing new to the topic under discussion. His treatment of the subject was anticipated by the younger Āryabhaṭa. The points of resemblance and difference in their treatment of this topic may be stated thus :—

Points of resemblance :

(i) Both Bhāskara and the younger Āryabhaṭa simplify the original equation $ax = by + c$ by means of the same preliminary operations.

(ii) Both of them take the zero as the *mati* and, therefore, do not require the last quotient used by the elder Aryabhaṭa. Consequently both stop the operations of mutual division when unity is reached as the remainder.

(iii) Both consider the same particular cases of the equation $ax = by + c$.

(iv) Both obtain the roots of the equation $ax = by \pm c$ from the roots of the equation $ax = by \pm 1$ by multiplying the latter roots by the *kṣepa* c.

(v) Both give similar rules for obtaining the least roots of the reduced equation $Ax = By + C$ by division by B and A.

(vi) Both give similar rules for finding general expressions for the values of the unknown quantities.

(vii) According to both of them a chain of quotients gives the roots of the equation $Ax = By + C$ or $Ax = By - C$ according as the number of quotients in the chain is even or odd.

(viii) Both give similar rules for deducing the roots of the equation $ax = by - c$ from those of the equation $ax = by + c$.

(ix) Both give similar rules for the solution of simultaneous indeterminate equations of the first degree in which the divisors are the same.

Points of difference:

(i) Bhāskara's first method of solution of the reduced equation $Ax = By + C$ is not giv. n by the younger Āryabhaṭa. In this method Bhāskara's chain of quotients, etc., although deduced from the elder Āryabhaṭa's chain, differs from the chain of the younger Āryabhaṭa. It would be interesting to compare these three chains.

Bhaskara's chain.	Chain of Aryabhata I.	Chain of Aryabhata II.
Q_1	Q_1	Q_1
Q_2	Q_2	Q_2
Q_3	Q_3	Q_3
...	...	...
...	...	...
...	...	...
Q_{n-2}	Q_{n-2}	Q_{n-2}
Q_{n-1}	Q_{n-1}	Q_{n-1}
C	$Q_n t + C$	1 [$\because C = 1$]
0	t	~~0~~
~~C~~	C	~~1~~

Bhāskara rejects the last C as superfluous and the younger Āryabhaṭa the last two numbers 0 and 1 for the same reason.

Bhāskara's chain gives the roots of the equation $Ax = By + C$ or $Ax = By - C$ and the younger Āryabhaṭa's chain those of the equation $Ax = By + 1$ or $Ax = By - 1$.

(ii) Bhāskara does not reject the case when the *kṣepa* is negative and divisible by the divisor. In the equation

$$Ax = By - mA$$

$$x = -m \text{ when } y = 0.$$

Bhāskara takes $x = Bt - m$ and $y = At$ and gives such numerical values to t as will make x positive.

(iii) Bhāskara gives a rule for the solution of the general simultaneous indeterminate equations of the first degree. This rule is as follows:

" हारे विभिन्ने गुणके च भिन्ने स्यादाद्यराशेर्गुणकस्तु साध्यः ।
द्वितीयभाज्यघ्नतदाद्यजो गुणः क्षेपो भवेत् क्षपयुतो द्वितीये ॥
द्वितीयभाज्यघ्नतदाद्यहारो भाज्यो भवेत्तत्र हरो हरः स्यात् ।
एवं प्रकल्प्यापि च कुट्टकेऽथ जातो गुणश्चाद्यहरेण निघ्नः ॥
गुणो भवेदाद्यगुणेन युक्तो हरघ्नहारोऽत्र हरः प्रदिष्टः ।
अथ तृतीयेऽपि तथैव कुर्यादेवं वहूनामपि साधयेत्तु ।" [26]

The following translation and *rationale* of the above rule, given by the present writer, appeared in the *Bulletin of the Calcutta Mathematical Society*, Vol. XVII:

" When the given divisors and multipliers are different in the various conditions—each condition giving rise to an indeterminate equation—to be satisfied in a complex problem on *kuṭṭaka* (*i.e.*, when the required integer has to satisfy various conditions leading to indeterminate equations of the first degree), an integer satisfying the first condition should, first of all, be found. The integer thus obtained, being multiplied by the (known) dividend in the second condition and then increased by the *kṣepa* of the second condition, becomes the *kṣepa* of a new (*lit.*, second) problem (the first condition being taken as the first problem), the (known) dividend (*i.e.*, multiplier) being the product of the divisor of the first condition and dividend of the second condition, and the divisor being the same as the divisor of the second condition. With these *kṣepa*, dividend and divisor, a multiplier should be found by the rule of *kuṭṭaka*. The product of this multiplier and the divisor of the first condition, being increased by the (previously unknown) multiplier satisfying the first condition (as originally obtained), is the required integer satisfying the first two conditions. (If many values of the common multiplier be desired or the *kṣepas* be negative or there be three conditions in the problem), proceed similarly taking, as instructed by previous teachers, the product of the divisors of the first two conditions as the divisor. Proceed similarly if there be more than three conditions in the problem."

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[26] *Bulletin*, Vol. XVII, p. 91.

Let the equations be

$$a_1x + c_1 = b_1y \qquad \dots \text{(i)}$$
$$a_2x + c_2 = b_2z \qquad \dots \text{(ii)}$$
$$a_3x + c_3 = b_3w \qquad \dots \text{(iii)}$$

Let $x = \alpha$ satisfy (i). Then $x = \alpha + b_1t$ (where t is any integer) will also satisfy (i). Let us suppose that this value of x also satisfies (ii). Then we must have

$$a_2(\alpha + b_1t) + c_2 = b_2z$$
or
$$a_2b_1t + (a_2\alpha + c_2) = b_2z.$$

This is the equation corresponding to the new problem referred to in the rule. If $t = \tau$ satisfies this equation, the value of x which satisfies (i) and (ii) is $\alpha + b_1\tau$ as given by the rule.

If $x = \beta$ satisfy (i) and (ii),

$$b_2(a_1\beta + c_1) = b_1b_2y$$
and
$$b_1(a_2\beta + c_2) = b_1b_2z.$$
Therefore
$$(a_1b_2 + a_2b_1)\beta + b_2c_1 + b_1c_2 = b_1b_2(y + z)$$

This shows that $x = \beta$ also satisfies the equation

$$(a_1b_2 + a_2b_1)x + b_1c_2 + b_2c_1 = b_1b_2(y + z) \qquad \dots \text{(iv)}$$

Hence the value of x, which satisfies (i), (ii), and (iii) also satisfies equations (iv) and (iii). Thus, the solution of the three equations (i), (ii), and (iii) is reduced to the solution of the simultaneous equations, namely (iv) and (iii). A solution of (iv) has already been obtained, the value of x being $\alpha + b_1\tau$. From this we can find a value of x which will satisfy another equation also, namely, equation (iii). In this case the divisor in the first, namely (iv), of this new couple of equations (iv) and (iii), is b_1b_2 which is the product of the divisors in the first two equations. Hence the rule *haraghnahāro = tra haraḥ pradiṣṭaḥ* for three conditions in which the *kṣepas* are all positive. We are not concerned with the dividend and *kṣepa* in equation (iv), as a solution has already been obtained without them.

When the *kṣepas* are negative, the equations become

$$a_1x - c_1 = b_1y \qquad \dots \text{(v)}$$
$$a_2x - c_2 = b_2z \qquad \dots \text{(vi)}$$
$$a_3x - c_3 = b_3w \qquad \dots \text{(vii)}$$

Equations (v) and (vi) may be written

$$a_1 (b_1 b_2 - x) + c_1 = b_1 (a_1 b_2 - y)$$

$$a_2 (b_1 b_2 - x) + c_2 = b_2 (a_2 b_1 - z)$$

i.e.,

$$a_1 x' + c_1 = b_1 y' \qquad \ldots \quad \text{(viii)}$$

$$a_2 x' + c_2 = b_2 z' \qquad \ldots \quad \text{(ix)}$$

where

$$x' \equiv b_1 b_2 - x.$$

Hence the solution of equations (v) and (vi) can be made to depend on the solution of equations (viii) and (ix) which are the same as equations (i) and (ii).

Because $x' = b_1 b_2 - x$,

$\therefore$ $x = b_1 b_2 - x'$.

Hence the rule for the case in which the *ksepas* are negative. Similarly the rule for the solution of three equations with negative *ksepas* might be deduced.

Enough has been written to show that all the different Indian methods of solution of the equation $ax = by + c$ have grown out of the elder Āryabhaṭa's method which follows, by simple and natural steps, from the first four simple rules and does not depend on Euclid's method of finding the greatest common divisor of two numbers or of proving by that process that two numbers have no common factor.

The Equation ax + by = c.

We have hitherto considered the equation $ax = by + c$ or $ax - by = c$ which is more important than the equation $ax + by = c$. The Chinese problems of "hundred hens" and a similar problem given by Mahāvīra lead ultimately to equations of the form $ax + by = c$. Mahāvīra's problem might or might not have been suggested by the Chinese problems. But it is extremely doubtful if Mahāvīra's rule for the solution of such problems was borrowed from China. Chang's rule for the solution of his problem of "hundred hens" does not seem to have been understood by Chinese scholars. There are two problems of hundred hens which lead to the following two sets of equations:

(i) $x + y + 3z = 100,$

$5x + 3y + z = 100.$

(ii) $x + y + 4z = 100,$

$5x + 4y + z = 100.$

These two sets lead respectively to the indeterminate equations :

$$7x = 4\,(25 - y),$$
$$19x = 15\,(20 - y),$$

the solutions of which can be easily found by inspection. Mahāvīra's problem leads to the equations

$$3x + 5y + 7z + 9w = 100,$$
$$5x + 7y + 9z + 3w = 100.$$

Integral solutions of these two equations are not so easily obtained as those of the equations which arise from the Chinese problems. Mahāvīra has given a rule for the general case which will be found on pages 133 and 134 of Rangāchārya's Translation of the *Ganitasāra-saṃgraha* .

For further elucidation of the rules explained above let me solve the equation

$$63x = 100y + 90$$

by the different Indian rules.

Here the divisor is 63, the dividend is 100, and the *kṣepa* is 90.

SOLUTIONS.

I. By the elder Āryabhaṭa's rule.

In this case the elder Āryabhaṭa's chain is given below :

1	63	100	1
2	26	37	1
1	4	11	2
	1	3	3
		0	

$1 \rightarrow 100t + 2430$
$1 \rightarrow (37t + 900) + (26t + 630) = 63t + 1530$
$1 \rightarrow (26t + 630) + (11t + 270) = 37t + 900$
$2 \rightarrow 2(11t + 270) + (4t + 90) = 26t + 630$
$2 \rightarrow 2(4t + 90) + (3t + 90) = 11t + 270$
$1 \rightarrow (3t + 90) + t = 4t + 90$
$3 \rightarrow 3t + 90$
t
90

Now $63t + 1530 \equiv 18$ (mod. 63).

$\therefore$ the least value of $y = 18$;

Again, $100t + 2430 \equiv 30$ (mod. 100),

$\therefore$ the least value of $x = 30$.

N. B.—The quotients of division by 63 and 100 should be the same.

II. By Brahmagupta's rule and Mahāvīra's first rule.

(*a*) *When the number of quotients is even.*

After four quotients, *viz.*, 1, 1, 1, 2, have been obtained, the divisor becomes 4, and the dividend becomes 11. The *kṣepa* is 90.

At this stage we find, by inspection, that $11 \times 2 + 90$ is divisible by 4, the quotient being 28. Hence the chain in this case becomes (on rejection of the first quotient)

$$\begin{aligned} 1 &\rightarrow 1 \times 86 + 58 = 144 \\ 1 &\rightarrow 1 \times 58 + 28 = 86 \\ 2 &\rightarrow 2 \times 28 + 2 = 58 \\ 28 & \\ 2 & \text{ (\textit{mati} or assumed number)} \end{aligned}$$

Now $144 \equiv 18$ (mod. 63).

∴ the least value of y is 18.

∴ the least value of x is 30.

(*b*) *When the number of quotients is odd.*

After five quotients, *viz.*, 1, 1, 1, 2, 2, have been obtained, the divisor becomes 4, the dividend becomes 3, and the *kṣepa* 90 remains unchanged. Since the number of quotients in this case is odd, the sign of the *kṣepa* is changed by transposition so that the divisor is converted into the dividend and *vice versa*. Thus the new divisor becomes 3 and the new dividend becomes 4, and the new *kṣepa* becomes -90. The *mati* (assumed integer) is now taken to be 24, since $4 \times 24 - 90$ is divisible by 3, the quotient being 2.

Now we have the following chain after rejecting the first quotient

$$\begin{aligned} 1 &\rightarrow 1 \times 86 + 58 = 144 \\ 1 &\rightarrow 1 \times 58 + 28 = 86 \\ 2 &\rightarrow 2 \times 28 + 2 = 58 \\ 2 &\rightarrow 2 \times 2 + 24 = 28 \\ 2 & \\ 24 & \end{aligned}$$

As $144 \equiv 18$ (mod. 63), the least value of y is 18.

Therefore the least value of x is 30.

III. By Mahāvīra's second rule.

In this case the operation of mutual division is stopped when unity is reached as the remainder. Here the number of remainders is six, *i.e.*, even. Mahāvīra uses the elder Āryabhaṭa's chain, taking t equal to 1.

$$\therefore \quad y = 63 + 1530.$$

When this is reduced by 63, the least value of y is found to be 18. Hence the least value of x is 30.

IV. By the younger Āryabhaṭa's rule.

Removing the G. C. M. 9 from the divisor and the *kṣepa* and also the G. C. M. 10 of the dividend and the reduced *kṣepa* we have

the reduced divisor = 7,
the reduced dividend = 10,
and the reduced *kṣepa* = 1.

Thus we have the following chain of the younger Āryabhaṭa :

2	7	10	1
	1	3	

$$1 \rightarrow 1 \times 2 + 1 = 3$$
$$2$$
$$1$$

Therefore, with the reduced divisor, dividend, and *kṣepa* the quotient is 3 and the multiplier is 2.

Therefore, with the given divisor, dividend, and *kṣepa*

the quotient $(x) = 3 \times 10 = 30$
and the multiplier $(y) = 2 \times 9 = 18$

V. By Bhāskara's rules.

(*a*) In the elder Āryabhaṭa's chain take $t = 0$. Since the number of quotients obtained just before the unity-remainder is reached is six (*i.e.*, even), the two uppermost numbers given by the elder Āryabhaṭa's chain with $t = 0$ give a pair of values of x and y. Thus, $x = 2430$, $y = 1530$. Reducing these values by taking away 100 and 63 respec-

tively the same largest number (24 in the present case) of times, we get 30 and 18 as the least values of x and y respectively.

(*b*) Removing the highest common factor 10 from the dividend and *kṣepa*, we have

$$\text{divisor} = 63,$$

$$\text{reduced dividend} = 10,$$

and $$\text{reduced } k\d{s}epa = 9.$$

According to Bhāskara's rule we have the following chain:

6	63	10	0
	3	10	3
		1	

$$\begin{aligned} 0 &\rightarrow 0 \times 171 + 27 = 27 \\ 6 &\rightarrow 6 \times 27 + 9 = 171 \\ 3 &\rightarrow 3 \times 9 + 0 = 27 \\ 9 & \\ 0 & \end{aligned}$$

As the number of quotients is three (*i.e.*, odd), 27 and 171 are a pair of roots of the equation

$$63x = 10y - 9.$$

Now $27 \equiv 7 \pmod{10}$, and $171 \equiv 45 \pmod{63}$.

$\therefore$ 7 and 45 are the least positive integral roots of

$$63x = 10y - 9.$$

$\therefore$ the least positive integral roots of the equation

$$63x = 10y + 9$$

are $(10 - 7)$ and $(63 - 45)$, *i.e.*, 3 and 18.

$\therefore$ the least positive integral roots of the equation

$$63x = 100y + 90$$

are $$x = 3 \times 10 = 30, \ y = 18.$$

(*c*) Removing the G. C. M. 9 from the divisor and the *kṣepa* we have

the reduced divisor = 7,

the dividend = 100,

and the reduced *kṣepa* = 10.

According to Bhāskara's rule we have the following chain:

3	7	100	14
	1	2	

$$14 \rightarrow 14 \times 30 + 10 = 430$$
$$3 \rightarrow 3 \times 10 + 0 = 30$$
$$10$$
$$0$$

As the number of quotients is 2 (*i.e.*, even), 430 and 30 are a pair of roots of the equation

$$7x = 100y + 10.$$

Now, $430 \equiv 30 \pmod{100}$, and $30 \equiv 2 \pmod{7}$.

$\therefore$ 30 and 2 are the least roots of the above equation.

$\therefore$ the least roots of the original equation are

30 and 18 ($= 2 \times 9$).

(*d*) Removing the G. C. M. 9 from the divisor and the *kṣepa* and the G. C. M. 10 from the dividend and the reduced *kṣepa* we have

the reduced divisor = 7,

the reduced dividend = 10,

and the reduced *kṣepa* = 1.

As the reduced *kṣepa* is unity, the rest of the process is the same as in IV above.

From the above solutions it will be easy to see how the elder Āryabhaṭa's rule and Mahāvīra's second rule may be applied to an equation (say, $63x = 10y + 9$) whose dividend and divisor give an odd number of quotients by mutual division.

SARADAKANTA GANGULI.

Group Theory and the History of Logarithms.

It is obvious that a necessary and sufficient condition that the arithmetic series of the first order

$$\ldots\; a-2d,\; a-d,\; a,\; a+d,\; a+2d,\; \ldots$$

is a group with respect to addition, is that it includes the sum of at least one pair of its terms. When this condition is satisfied it must therefore involve 0, and it may be assumed without loss of generality that $a = 0$ while d has an arbitrary value. Similarly, a necessary and sufficient condition that the geometric series

$$\ldots\; \frac{a}{r^2},\; \frac{a}{r},\; a,\; ar,\; ar^2,\; \ldots$$

is a group with respect to multiplication is that it includes the product of at least one pair of its terms. When this condition is satisfied it must therefore involve 1, and it may be assumed without loss of generality that $a = 1$. The most general group arithmetic series and group geometric series may therefore be assumed to be respectively

$$\ldots\; -2d,\; -d,\; 0,\; d,\; 2d,\; \ldots$$

$$\ldots\; \frac{1}{r^2},\; \frac{1}{r},\; 1,\; r,\; r^2,\; \ldots$$

When a (1, 1) correspondence is established between these two group series in such a way that kd of the arithmetic series corresponds to the identity of the geometric series then it results directly that to the product of two terms of the latter series there corresponds the sum of the two corresponding terms of the former diminished by kd. Hence it follows that to the nth power of a term in the latter series there corresponds n times the corresponding term of the former diminished by $(n-1)\,kd$. In particular, when the identities of the two series correspond there result the ordinary rules for operating with logarithms, and we have an actual (1, 1) correspondence between the elements of the two groups and their products. That is, the two groups are simply isomorphic and hence represent the same abstract group.

In the early developments relating to logarithms by John Napier, Jobst Bürgi and others, a (1, 1) correspondence was established between an arithmetic series and a geometric series, and it was assumed that the product of any two terms of this geometric series is contained therein. Hence these investigators were dealing with groups without knowing it. In fact, the group concept had not then been explicitly formulated. One can, however, understand their work better now by considering it from the standpoint of group theory, especially since the groups involved therein are closely related to two of the most important abelian groups of the history of mathematics, *viz.*, the group of addition of the real numbers and the group of multiplication of the positive numbers.

The history of logarithms cannot be fully understood without considering the significance of the fact that neither John Napier nor Jobst Bürgi made the identity of the arithmetic series correspond to the identity of the geometric series in establishing a (1, 1) correspondence between two such group series. That their failure to do so was unwise results clearly from the fact that their tables were soon replaced by others in which the identities of these two series correspond to each other. In fact, John Napier himself realized after the publication of his tables that it was desirable to make these identities correspond and he suggested this desirability to H. Briggs in 1615 The correspondence which they selected had, however, two important advantages since they avoided thereby not only the need of fractions in using their tables but also the need of using negative numbers therein. Both of these advantages were naturally appreciated more highly in their times than later when these avoided numbers became more familiar to those for whom their tables were intended.

To understand the (1, 1) correspondence which John Napier and Jobst Bürgi established between two simply isomorphic group series it should be explicitly noted that while an arbitrary (1, 1) correspondence between the elements of two simply isomorphic groups can be established this correspondence does not represent an automorphism, or a simple

isomorphism of the groups concerned, unless the product of every pair of corresponding elements and the squares of the corresponding elements correspond therein. In particular, the identities of the two groups must correspond therein. As John Napier and Jobst Bürgi did not let these identities correspond in their tables it is not possible that the logarithm of the product of two numbers is equal to the sum of the logarithms of these numbers according to these tables and hence the question arises whether they should now be regarded as tables of logarithms, or tables of antilogarithms respectively.

From the preceding paragraph it results that if in a (1, 1) correspondence between an arithmetic series and a geometric series it is assumed that the sum and the product of the two terms in these respective series occur therein, and also that such sums and such products correspond therein whenever the addends correspond to the related factors, then we are dealing with two simply isomorphic group series. Hence the rules which govern the operations with group elements apply directly to these series. There are, however, still two degrees of freedom since the common difference of the arithmetic series and the ratio of the geometric series can be assigned arbitrary values except that the former cannot be 0 and the latter cannot be 1 since these are the identities of the groups concerned and the order of the groups is supposed to exceed 1.

Since a geometric group series must be composed of powers of the base the corresponding terms of the arithmetic group series must be the same multiples of the exponents of these powers when there is a simple isomorphism between the two group series. A necessary and sufficient condition that the former terms are actually the exponents of the corresponding elements of the geometric group series is that the base of the latter correspond to the unity in the arithmetic series, and a necessary and sufficient condition that unity appears in the arithmetic group series is that the common difference of this series is a unit fraction. In particular, when the base of the geometric series is 10 and corresponds to

unity in the arithmetic group series there results our common system of logarithms.

It should be emphasized that the most important step in the development of logarithms was not the use of the exponent of the base as logarithms but the establishment of a simple isomorphism between the arithmetic group series and the corresponding geometric group series. The ordinary rules of working with logarithms are secured thereby and the theory of logarithms is a little more general if the idea of exponent is not introduced into it since both the base of the geometric group series and the common difference of the corresponding arithmetic series are then not specialized On the other hand, the concept of logarithms as exponents has the advantage of connecting more directly the theory of exponents and the theory of logarithms. As an instance of a modern treatment of logarithms without the use of exponents we may cite the exposition of this subject by E. Borel, *Algèbre*, second cycle, 1921, page 380.

It may be noted in this connection that the functional equation

$$f(xy) = f(x) + f(y)$$

establishes essentially a (1, 1) correspondence between an arithmetic and a geometric group series in which the identities of the two groups correspond. Hence this functional equation can be studied advantageously from the standpoint of group theory and its connection with the subject of logarithms is thus clearly exhibited. This functional equation was considered by A. L. Cauchy in his *Analyse algébrique*, 1821, page 110, under the condition that $f(x)$ remains continuous between two arbitrary positive values of the variable x, and the conclusion is reached that $f(x) = a \log x$.

G. A. MILLER.

Solutions.

Question 1583.

(V. RAMASWAMI IYER, M.A.):—Let P be a point moving in the perpendicular bisector of the side BC of a triangle ABC. If M be the centre of the pedal circle of P, show that PA is parallel to MD where D is the middle point of BC.

Solution by K. Rangaswamy, S. Srinivasan, A. Govinda Wariar and N. K. Narasimhamoorthy.

Construction.—Produce AP to meet BC at X

Proof.—D, P, F, B are concyclic.

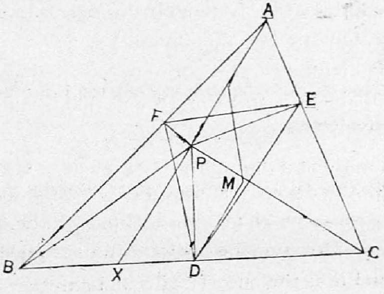
$\therefore \quad \angle PBD = \angle PFD = a$ (say)

E, P, F, A are concyclic.

$\therefore \quad \angle FAP = \angle FEP = c$ (say)

E, P, D, C are concyclic.

$\therefore \quad \angle PCD = \angle PED = a$ (because $P\hat{B}D = P\hat{C}D$).



Also $\quad P\hat{D}F = P\hat{B}F = b$ (say).

Now since $\quad$ M is the circum-centre of ΔDEF

$\therefore \quad \angle MDF = 90° - D\hat{E}F$

$= 90° - (a + c).$

$\therefore \quad \angle MDP = 90° - (a + b + c).$

$\therefore \quad \angle MDC = a + b + c.$

$= \angle AXC.$

$\therefore \quad$ AP is $\parallel$ to MD.

Also solved analytically by Hansraj Gupta.

Question 1585.

(N. K. NARASIMHA MURTHY):—Show that the digits (1,2,3, ... 9) can be divided in *only two* ways into two groups so that a number formed by one of the groups contains, when squared, the digits in the other group. Find these numbers.

Prove that the same digits cannot be formed into numbers so that the cube of one contains the digits of the other (each digit occurring once and only once.

Solution by the Proposer and by P. S. Narasimhaiah.

In the present problem we have to divide the digits (1, 2, 3, 4, 5, 6, 7, 8, 9) into two groups so that the number formed out of the digits of one group contains, when squared, the digits of the other group.

Remembering that any integer is congruent *modulo* 9 to the sum of its digits, we have to divide $1+2+3...+9=45$ into two parts x, y such that $x+y=45$ and $x^2 \equiv y$. (mod. 9) We therefore get pairs of numbers (8, 37) (9, 36) (17, 28) (18, 27) where $8^2 \equiv 1 \equiv 37$ (mod 9), etc. Now we have to divide the digits 1, 2, 3 ... 9, into two groups so that the sum of one group of digits is 8, 9, 17 or 18 and that of the other is 37, 36, 28 or 27 respectively.

The least number of 3 digits, 100, when squared, gives 10^4 a number of 5 digits and the least number of 4 digits, 1000, when squared, gives the least number of 7 digits. Hence the 9 digits can be divided into the required groups, one containing 3 and the other, 6 digits. Also $\sqrt{100000} > 316$ and thus the number of 3 digits to be formed should be greater than 316.

More than 4 pairs of numbers of the type (8, 37) do not occur. The least and the greatest of the sum of 3 of the digits 1, 2, 3, ... 9 being 6 and 24 respectively, the selection of the numbers is restricted to the range 6 to 24, and four only out of these satisfy the conditions.

Next we have to select groups of 3 digits so that their sum may be 8, 9, 17 or 18.

For the sum 8, we get 2 groups of digits, (1, 2, 5) or (3, 1, 4). A number must be formed out of these, if possible, to contain, when squared

the other digits, each once and only once. A necessary condition for this is that the number formed should not end in the digits 1, 5 or 6 and it must not be less than 316.

Satisfying these conditions the only number that can be formed from the group (1, 2, 5) is 512 but it has to be rejected as the last digit of the square is repeated for $12^2 = 144$. The number 413 that can be formed from the group (1, 3, 4) is rejected because when squared 413^2 begins with the digit 1 which it already contains.

For the sum of the 3 digits equal to 9, we get the groups (1, 2, 6) (1, 3, 5) (2, 3, 4) and the numbers greater than 316 that can be formed from these digits so as not to end in 1, 5 or 6 are 612, 513, 324, 342, 423, 432. All these are to be rejected due to some reason or other.

In the case of the sum equal to 17 the groups of digits are (1, 7, 9) (2, 6, 9) (2, 7, 8) (3, 5, 9) (3, 6, 8) (4, 5, 8) (4, 6, 7). Of these (4, 5, 8) gives the number 854 which when squared gives 729316, a number containing the other digits. Also, no other number satisfies the condition.

If 18 be taken as the sum of the digits the groups are (1, 8, 9) (2, 7, 9) (3, 6, 9) (3, 7, 8) (4, 5, 9) (4, 6, 8) (5, 6, 7). The number 567 formed from the last set is such that $567^2 = 321489$, a number containing the remaining digits. No other number is possible.

Thus only two numbers, 854 and 567 give, when squared, numbers containing the remaining digits.

The least number of 3 digits *viz*, 100 gives when cubed, 10^6 the least number of 7 digits and the greatest number of 2 digits, 99 contains when cubed, 6 digits. Thus the sum of the number of digits of any number and that of its cube can never be 9. Therefore the digits 1, 2, 3, ... 9 cannot be divided into two groups so that a number formed from the digits of one group may contain the digits in the other group when cubed.

[The first part of this problem is a modification of Problem 22, Page 79 in Mr. A. A. Krishnaswamy Iyengar's "Mathematical Problem Papers"].

Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their questions.

1606. (R. RAMANUJAN) :—ABC is an acute-angled triangle whose orthocentre is O. D, E, F are the feet of the altitudes from A, B, C respectively. BA and DE meet at G, CA and DF at H, BC and HG at Q. L, M, P are the middle points of AB, AC and BC respectively.

D′, E′, F′, G′ ... etc. are similar points with respect to another acute-angled triangle A′B′C′.

Given that

(1) $1\frac{1}{4}(AB^2 + AC^2 + BC^2) + OD^2 + OE^2 + OF^2$
$+ GL^2 + HM^2 + PQ^2 - FG^2 - HE^2 - DQ^2 - 12R^2$
= the similar expression with respect to the triangle A′B′C′. where R represents the circum-radius of the triangle ABC.

(2) one side of the triangle DEF = 11, and the median bisecting it = 17.

(3) one side of the triangle D′E′F′ = 5, and the median bisecting it = 19,

find the ratio the perimeters of the triangles DEF and D′E′F′.

1607. (R. RAMANUJAN) :—Prove the theorem of Pythogoras without constructing squares on the sides of the triangle.

1608. (R. RAMANUJAN) :—Construct a triangle (geometrically), given the base, vertical angle and the internal or external bisector of one of the base angles

1609. (R. RAMANUJAN) :—Construct a triangle (geometrically), given the base, one of the base angles and the internal or external bisector of the vertical angle.

1610 (S MUKHOPADHYAYA):—ABC is any triangle. BX and CX are, respectively, the internal (external) trisectors of the angles B and C, nearest to BC. CY and AY are, respectively, the internal (external) trisectors of the angles C and A, nearest to CA. AZ and BZ are, respectively, the internal (external) trisectors of the angles A and B, nearest to AB. Prove that the triangle XYZ is equilateral, by elementary geometry.

JL
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LIST OF JOURNALS & BOOKS RECEIVED IN THE LIBRARY

during the months of December and January 1931.

1 Acta Mathematica, **57**, 3, 4.

2 American Mathematical Monthly, **38**, 9, Part I.

3 Astrophysical Journal, **74**, 3, 4.

4 Bulletin of the American Mathematical Society, **37**, 10, 11.

5 Bulletin of the Calcutta Mathematical Society, **23**, 3.

6 Bulletin des Sciences Mathematiques, **55**, Nov. 1931.

7 Crelle's Journal, **166**, 1, 2.

8 Mathematical Gazette, **15**, 216 (3 copies).

9 Philosophical Magazine, **12**, 180, 81.

10 Philosophical Transactions of the Royal Society, **230**, 323 to 362, 363—385.

11 Proceedings of the Physico-Mathematical Society of Japan, **13**, 8, 10.

12 Proceedings of the Royal Society of London, **134**, 823, 824.

13 Publication de la Faculte des Sciences de l'University Masaryk.

1 Nova Metoda Mereni Radiace Gatek Radioaktivnich (L'action de Force des substances radioactives sur La Balance de Coulomb).

2 Kmity Shrazenych Sonstav (Les Vibrations des systems couples).

3 Reseaux R. A. Invariants Egaux.

4 Trois Theoremes sur l'Homologie.

14 Quarterly Journal of Mathematics, **2**, 8.

15 Transactions of the Royal Society of South Africa, **20**, 1.

Books.

1 Bibliography of Projective Differential Geometry, by Pauline Sperry.

2 Sydney University Reprints, Ser. 11, Vol. 1, Nos. 20—25.

Numbers in black type refer to the volumes, and those in ordinary type to the numbers of the issues.

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